Log transformed confidence intervals using Kaplan-Meier estimator

$g\left(S\left(t\right)\right)=\ln\left(-\ln\left(S\left(t\right)\right)\right)$ and consequently $S(t)=\exp\left(-\exp\left(g\left(S\left(t\right)\right)\right)\right)$

- $g\left(S\left(t\right)\right)$ is estimated using $g\left(S_n\left(t\right)\right)=\ln\left(-\ln\left(S_n\left(t\right)\right)\right)$ and

  $\hat{\operatorname{var}}\left(g\left(S_n(t)\right)\right)\approx\dfrac{\hat{\operatorname{var}}\left(S_n\left(t\right)\right)}{\left(S_n\left(t\right)\times\ln\left(S_n\left(t\right)\right)\right)^2}$ and

  $\sqrt{\hat{\operatorname{var}}\left(g\left(S_n(t)\right)\right)}\approx-\dfrac{\sqrt{\hat{\operatorname{var}}\left(S_n\left(t\right)\right)}}{S_n\left(t\right)\times\ln\left(S_n\left(t\right)\right)}$ as $S_n\left(t\right)\times\ln\left(S_n\left(t\right)\right)<0$

- Approx. conf. Int. for $g\left(S\left(t\right)\right)$ is given by

  $\left(g\left(S_n(t)\right)-z_{\alpha/2}\sqrt{\hat{\operatorname{var}}\, g\left(S_n(t)\right)};\ g\left(S_n(t)\right)+z_{\alpha/2}\sqrt{\hat{\operatorname{var}}\, g\left(S_n(t)\right)}\right)$ i.e.

  $\left(g\left(S_n(t)\right)+z_{\alpha/2}\dfrac{\sqrt{\hat{\operatorname{var}}\left(S_n(t)\right)}}{S_n(t)\times\ln S_n(t)};\ g\left(S_n(t)\right)-z_{\alpha/2}\dfrac{\sqrt{\hat{\operatorname{var}}\left(S_n(t)\right)}}{S_n(t)\times\ln S_n(t)}\right)$ as

  $\sqrt{\hat{\operatorname{var}}\, g\left(S_n(t)\right)}=-\dfrac{\sqrt{\hat{\operatorname{var}}\left(S_n(t)\right)}}{S_n(t)\times\ln S_n(t)}$

Now, we will apply the inverse transform $S(t)=\exp\left(-\exp\left(g\left(S\left(t\right)\right)\right)\right)$ to get a confidence interval for $S(t)$ as requested. Note that the inverse transform is a decreasing function and consequently the inverse transform of the lower bound will be the upper bound of the new CI ($\Psi(x)=\exp(-\exp(x))$ and $\Psi'(x)=-e^x\,\Psi(x)<0$).

So, the upper bound of the new Confidence interval is the inverse transform of $g\left(S_n(t)\right)-z_{\alpha/2}\sqrt{\hat{\operatorname{var}}\, g\left(S_n(t)\right)}$.

$$
\begin{aligned}
UB &= \exp\left(-\exp\left(g\left(S_n(t)\right) - z_{\alpha/2}\sqrt{\hat{\text{var}}\left(g\left(S_n(t)\right)\right)}\right)\right) \\
&= \exp\left(-\exp\left(g\left(S_n(t)\right) + z_{\alpha/2}\frac{\sqrt{\hat{\text{var}}\, S_n(t)}}{S_n(t)\times \ln S_n(t)}\right)\right) \\
&= \exp\left(-\exp\left(g\left(S_n(t)\right)\right)\times \exp\left(z_{\alpha/2}\frac{\sqrt{\hat{\text{var}}\, S_n(t)}}{S_n(t)\times \ln S_n(t)}\right)\right) \\
&= \exp\left(-\exp\left(g\left(S_n(t)\right)\right)\times U\right) \qquad \text{defining } U = \exp\left(z_{\alpha/2}\frac{\sqrt{\hat{\text{var}}\, S_n(t)}}{S_n(t)\times \ln S_n(t)}\right) \\
&= \exp\left(-\exp\left(g\left(S_n(t)\right)\right)\right)^U \\
&= \left(S_n(t)\right)^U
\end{aligned}
$$

And for the lower bound

$$
\begin{aligned}
LB &= \exp\left(-\exp\left(g\left(S_n(t)\right) + z_{\alpha/2}\sqrt{\hat{\text{var}}\left(g\left(S_n(t)\right)\right)}\right)\right) \\
&= \exp\left(-\exp\left(g\left(S_n(t)\right) - z_{\alpha/2}\frac{\sqrt{\hat{\text{var}}\, S_n(t)}}{S_n(t)\times \ln S_n(t)}\right)\right) \\
&= \exp\left(-\exp\left(g\left(S_n(t)\right)\right)\times \exp\left(-z_{\alpha/2}\frac{\sqrt{\hat{\text{var}}\, S_n(t)}}{S_n(t)\times \ln S_n(t)}\right)\right) \\
&= \exp\left(-\exp\left(g\left(S_n(t)\right)\right)\times \frac{1}{U}\right) \qquad \text{as } U = \exp\left(z_{\alpha/2}\frac{\sqrt{\hat{\text{var}}\, S_n(t)}}{S_n(t)\times \ln S_n(t)}\right) \\
&= \exp\left(-\exp\left(g\left(S_n(t)\right)\right)\right)^{1/U} \\
&= \left(S_n(t)\right)^{1/U}
\end{aligned}
$$

To sum-up:

- Compute $U = \exp\left(z_{\alpha/2}\frac{\sqrt{\hat{\text{var}}\, S_n(t)}}{S_n(t)\times \ln S_n(t)}\right)$
- The log-transformed CI is $\left(S_n(t)^{1/U}; S_n(t)^U\right)$